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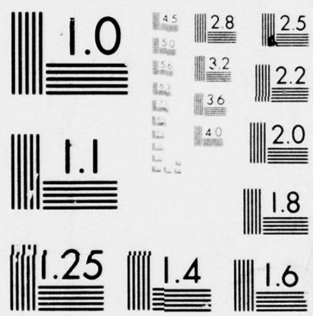

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THINNING OF A POINT PROCESS

OVER TIME

BY

Richard F. Serfozo

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Thinning of a point process refers to the procedure in which points are randomly placed in a region and then they are deleted according to some rule. The aim is to answer questions such as (1) how can the random placement and detection of points be described mathematically? (2) what types of thinned processes arise from various thinning rules? (3) how much thinning is needed for a desired rarefaction of points? and (4) when does one reach diminishing returns in thinning? Examples of thinning procedures are debugging of computer programs and complex systems, filtration of particles from a solution, and the elimination of undesirable

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20. Abstract

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cell growth, insects or plants. This paper addresses several thinnings in which points are deleted over time. We show how the asymptotic behavior of a thinned process is equivalent to that of extreme values of the lives of its points under the thinning. We use this to describe independent, regenerative, and semi-stationary thinnings.

Abstract

Thinning of a point process refers to the procedure in which points are randomly placed in a region and then they are deleted according to some rule. The aim is to answer questions such as (1) how can the random placement and detection of points be described mathematically? (2) what types of thinned processes arise from various thinning rules? (3) how much thinning is needed for a desired rarefaction of points? and (4) when does one reach diminishing returns in debugging? Examples of thinning procedures are debugging of computer programs and complex systems, filtration of particles from a solution, and the elimination of undesirable cell growth, insects or plants. This paper addresses several thinnings in which points are deleted over time. We show how the asymptotic behavior of a thinned process is equivalent ot that of the extreme values of the lives of its points under the thinning. We use this to describe independent, regenerative, and semi-stationary thinnings.

Thinning of a Point Process Over Time

by

Richard F. Serfozo, Syracuse University

1. Introduction

The first study of thinnings of point processes, Renyi (1956), contains the following result. Suppose that points are placed on the nonnegative real line R_+ such that the interpoint distances are independent and identically distributed with mean c. These points are thinned over time (or in stages) such that at time n each point that has survived till then is independently retained with probability p_n and deleted with probability $1 - p_n$. Let $N_n(t)$ denote the number of points still remaining in the interval $[0, a_n t]$ where $a_n = (p_1 \ldots p_n)^{-1} \to \infty$. Think of N_n as the thinned process at time n with a_n as a new unit of scale for R_+. Then N_n converges in distribution as $n \to \infty$ to a Poisson process with intensity c^{-1}.

As an illustration of this result, consider a production system in which parts (of one type) are produced and then sent through a series of n work stations. Defective parts occur according to a renewal process, and each one of these will be detected and discarded with probability $1 - p_m$ at the m-th station. Consequently, the flow of defective parts is thinned over n stages. If n is large so that $p_1 \ldots p_n$ is very small, then according to Renyi's result, the flow of defective parts from the n-th station can be approximated by a Poisson process.

Since 1956 twenty-four articles (see the references) have been written on various aspects of the following.

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Thinning Problem. Points are placed in a space according to a certain probability law, and then the points are randomly deleted according to some rule. Is it possible to normalize the thinned process by a rescaling of the space (as above with the a_n) so that it converges in distribution to some process as the thinning becomes complete? And what are the possible norming constants and limits?

This problem is analogous to the central limit problem for sums of random variables, which is concerned with finding norming constants A_n and B_n such that $B_n^{-1} \Sigma_{k=1}^{n} X_k - A_n$ (or its related stochastic process [3, Theorem 10.1]) converges in distribution. Here the $\Sigma_{k=1}^{n} X_k$ or its variance is converging to infinity, and in thinning, the thinned process is converging to zero. The rescaling by B_n is comparable to rescaling the space in a thinning. The central limit theory has yielded many insights into random phenomena in nature that arise as sums of a large number of random variables. For example, we can now readily identify various types of random phenomena that can be modeled as (or approximated by) normal or infinitely divisible random variables, Weiner processes, stable processes, etc. A knowledge of thinning will yield similar insights into point processes that arise from thinning procedures such as the following: (1) debugging of complex computer programs, weapons systems, communications networks etc. (a check-out routine describes a random path through the system and the residual errors form the thinned process), (2) filtration of pollutants in the air or water by magnetized smoke stacks, car mufflers or charcoal beds, (3) elimination of undesirable insect or animal pests, plants or cell growth, and (4) cutting down on potential disease carriers by an immunization program.

In a recent study [28] we showed how many thinnings of point processes can be analyzed in terms of compositions and inverses of random measures. In doing so we unified many of the results on thinnings. The main emphasis in [28], similar to most of the studies referenced herein, was on a single, very thorough, thinning

operation. In this article we consider thinnings in which points are deleted over time or by a sequence of thinning operations. In Section 3 we show that the asymptotic behavior of a thinned process is essentially equivalent to that of the extreme values (high level exceedances) of the lives of its points under the thinning. We use this result in Section 4 to describe independent, regenerative, and semi-stationary thinnings of point processes on R_+. Then in Section 5 we describe independent thinnings on more general spaces.

2. Notation.

We shall use the following notation for point processes and random measures. This is similar to that in the basic works [1], [9] and [13].

Let M denote the set of measures on the Borel sets $\mathcal{B}_+$ of R_+. that are finite on bounded sets. We endow M with the vague topology [1]. This topology is metrizable and the following are equivalent statements.

(i) $\mu_n \to \mu$ in the vague topology.

(ii) $\int f(x)d\mu_n(x) \to \int f(x)d\mu(x)$ for each continuous f on R_+ with compact support.

(iii) $\mu_n(a,b] \to \mu(a,b]$ for all a, b such that $\mu(\{a,b\}) = 0$.

Let $\mathcal{M}$ be the smallest σ-field on M containing the open sets of the vague topology. This is the same as the smallest σ-field that makes the mappings $\mu \to \mu(A)$, for $A \in \mathcal{B}_+$, measurable.

A random measure ξ on R_+ is defined to be a measurable mapping from a probability space to $(M, \mathcal{M})$. We denote by $\xi(A)$ the random variable describing the mass in the set $A \in \mathcal{B}_+$, and we let $\xi(t) = \xi([0,t])$ for $t \in R_+$. If $\xi(t)$ is integer-valued for each t, then ξ is called a point process.

A sequence ξ_n of random measures converges in distribution to a random measure ξ, written $\xi_n \xrightarrow{D} \xi$, if the distribution of ξ_n converges weakly to the distribution of ξ. That is, $Eh(\xi_n) \to Eh(\xi)$ for each bounded continuous function h on M, see [3].

The following are equivalent statements:

(i) $\xi_n \xrightarrow{\mathcal{D}} \xi$.

(ii) $\int f(x)d\xi_n(x) \xrightarrow{\mathcal{D}} \int f(x)d\xi(x)$ for each continuous f on R_+ with compact support.

(iii) $(\xi_n(A_1),\ldots,\xi_n(A_k)) \xrightarrow{\mathcal{D}} (\xi(A_1),\ldots, \xi(A_k))$ for any bounded intervals $A_1,\ldots,$ A_k in R_+ satisfying $\xi(\delta A_1) = \ldots = \xi(\delta A_k) = 0$ a.s., where δA denotes the boundary of A.

The above terminology also holds with obvious modifications for measures on locally compact second countable Hausdorff spaces such as R^n.

Much of our analysis will deal with compositions of measures. If ξ and η are random measures on R_+ then their composition $\xi o\eta$ is a random measure with $\xi o\eta(t) = \xi(\eta(t))$. We also let ξoc, for $c\varepsilon R_+$, denote the measure with $\xi oc(t) = \xi(ct)$. In the following result the ξ's and η's are random measures on R_+ and the α's and β's are random variables.

Lemma 2.1. (i) If $(\xi_n,\eta_n) \xrightarrow{\mathcal{D}} (\xi,\eta)$, where $\xi(t)$ is continuous a.s. or $\eta(t)$ is strictly increasing and $\eta(0) = 0$ a.s., then $\xi_n o\eta_n \xrightarrow{\mathcal{D}} \xi o\eta$.

(ii) If $(\xi_n,\alpha_n,\beta_n) \xrightarrow{\mathcal{D}} (\xi,\alpha,\beta)$, where $\alpha_n \le \beta_n$ and $\xi(\{\alpha,\beta\}) = 0$ a.s., then $\xi_n((\alpha_n,\beta_n]) \xrightarrow{\mathcal{D}} \xi((\alpha,\beta])$.

Proof. These assertions follow from Theorem 3.2 and Corollary 2.4 in [28].

3. Thinning and High-Level Exceedances of Stochastic Processes

The thinning procedure that we shall consider here and in the next section is as follows. Points are placed in R_+ at random sites $0 \le S_1 < S_2 < \ldots$ where $S_k \to \infty$ a.s. Throughout Sections 3 and 4 we assume the following.

General Assumption. $k^{-1}S_k \xrightarrow{\mathcal{D}} c$ where c is a positive constant.

The points are deleted over time (or in stages) according to some rule. We let Y_k denote the random length of time - which we take to be discrete - that the point at site S_k survives under the thinning. (In this section we make no assumptions on the dependency between the Y's or S's.) Then at time n, the

number of points still remaining in a set $A \in \mathcal{B}_+$ is given by

$$\xi_n(A) = \sum_k I(Y_k > n)\delta_{S_k}(A)$$

where $I(Y_k > n) = 1$ or 0 according as $Y_k > n$ or $Y_k \leq n$, and $\delta_s(\cdot)$ denotes the Dirac measure with unit mass at s. In this section we show how the asymptotic behavior of the thinned process ξ_n is related to large values of the Y's. In the next section we present more specific results when the Y's are a regenerative or semi-stationary sequence.

To record large values of the Y's we shall use the process

$$\eta_n(t) = \sum_{k=1}^{[t]} I(Y_k > n) \qquad t \in R_+$$

where $[t]$ is the integer part of t. The η_n is the exceedance process at level n associated with the Y's. Namely, $\eta_n(k)$ is the number of the lifetimes $Y_1, \ldots, Y_k$ which exceed n.

The following result shows that the convergence of ξ_n is equivalent to the convergence of η_n. Note that the randomness of the S's affects the convergence of ξ_n only through the constant c. Here and below the a_n and b_n are positive constants with $a_n \to \infty$, and ξ and η are random measures.

Theorem 3.1. A necessary and sufficient condition for $b_n^{-1}\xi_n \circ a_n \xrightarrow{\mathcal{D}}$ some ξ is that $b_n^{-1}\eta_n \circ a_n \xrightarrow{\mathcal{D}}$ some η. In either case $\xi \overset{\mathcal{D}}{=} \eta \circ c$.

Proof. Clearly

$$\xi_n(t) = \sum_{k=1}^{\zeta(t)} I(Y_k > n) = \eta_n(\zeta(t))$$

where $\zeta(t) = \max\{k : S_k \leq t\}$. Also note that the assumption $k^{-1}S_k \xrightarrow{\mathcal{D}} c$ is equivalent to $t^{-1}\zeta(t) \xrightarrow{\mathcal{D}} c^{-1}$. Thus the assertion follows by [28, Theorem 4.1].

Remark 3.2. Theorem 3.1 also holds if convergence in $\mathcal{D}$ is replaced throughout by convergence a.s., or convergence in probability. This follows from the nature of its proof.

4. *Stationary, Regenerative, and Semi-stationary Thinnings*

In this section we describe the asymptotic behavior of the thinned process $\xi_n = \sum_k I(Y_k > n)\delta_{S_k}$, as described above, for several types of thinnings. We label these thinnings according to the dependency structure on the Y's.

We begin with an independent thinning.

Theorem 4.1. Suppose $Y_1, Y_2, \ldots$ are independent with a common distribution. Then $\xi_n o a_n \xrightarrow{\mathcal{D}} \xi$, where ξ is a Poisson process with intensity λc^{-1}, if and only if

$$a_n P(Y_1 > n) \to \lambda > 0.$$

Proof. This is a special case of Theorem 4.4 below.

Example 4.2. (Renyi's result) Suppose the thinning is such that at time n each point that has survived is independently deleted with probability $1 - p_n$. That is $P(Y_1 > n) = p_1 \ldots p_n$. If $a_n = (p_1 \ldots p_n)^{-1} \to \infty$, then $\xi_n o a_n$ converges in distribution to a Poisson process with intensity c^{-1}.

Our next result is for stationary thinnings. In it we use the following.

Dependency Conditions 4.3. Let X_{nk} $(k, n \geq 1)$ be nonnegative integer-valued random variables.

(i) For any integers $1 \leq k_1 < k_2 \ldots < k_p < \ell_1 \ldots < \ell_q \leq a_n$, with $\ell_1 - k_p \geq i$

$$\begin{aligned} | \, & P(X_{nk_1} = \ldots = X_{nk_p} = X_{n\ell_1} = \ldots = X_{n\ell_q} = 0) \\ & -P(X_{nk_1} = \ldots = X_{nk_p} = 0)\, P(X_{n\ell_1} = \ldots = X_{n\ell_q} = 0) \, | \leq \alpha_{ni} \end{aligned}$$

where α_{ni} is nonincreasing in i and $\alpha_{ni_n} \to 0$ for some sequence $i_n \to \infty$ such that $i_n/a_n \to 0$.

(ii) $\lim_{n\to\infty}\sup a_n \sum_{j=2}^{a_n} P(X_{mn,1} \geq 1,\ X_{mn,j} \geq 1) = o(m^{-1})$ as $m \to \infty$.

These dependency conditions are essentially those used in [16] and [17] (the n and u_n in these references are our a_n and n respectively). Note that (i) is slightly weaker than the usual mixing conditions that describe dependencies.

Theorem 4.4. Suppose that $Y_1, Y_2, \ldots$ is a strictly stationary sequence such that the $X_{nk} = I(Y_k > n)$ satisfy Conditions 4.3 and

$$a_n P(Y_1 > n) \to \lambda > 0.$$

Then $\xi_n o a_n \xrightarrow{\mathcal{D}} \xi$ where ξ is a Poisson process with intensity λc^{-1}.

Proof. This is a special case of Theorem 4.7 below.

We now consider regenerative thinnings. Namely, we shall assume that $Y_1, Y_2, \ldots$ is a regenerative process over the integer-valued indices $1 = \tau_0 < \tau_1 < \tau_2 \ldots$ That is, the vectors

$$(\tau_k - \tau_{k-1}, Y_{\tau_{k-1}}, \ldots\ Y_{\tau_k - 2}, Y_{\tau_k - 1}, 0, 0, \ldots) \quad \text{for } k \geq 1$$

are independent and identically distributed. This might occur when the points are thinned in independent groups of random sizes. The above vector then would describe the lives of the points in the k-th group which consists of $\tau_k - \tau_{k-1}$ points.

As an example, suppose the initial points at sites $S_1, S_2, \ldots$ have generic makeups $\beta_1, \beta_2, \ldots$ which determine their lifetimes $Y_1, Y_2, \ldots$ such that

$$P(Y_k = n \mid \beta_k = i, \beta_\ell, Y_\ell (\ell \neq k)) = P(Y_1 = n \mid \beta_1 = i).$$

Assume that β_k is a positive recurrent Markov chain with a countable state space and $\beta_1 = i$. Let $1 = \tau_0 < \tau_1 < \ldots$ be the successive indices k for which $\beta_k = i$. Then Y_k is a regenerative process over the τ's. Another interpretation of this example is that the initial points are all the same, but the thinning is inhomogeneous in that at site S_k it depends on the parameter β_k.

For the next result we assume that $Y_1, Y_2, \ldots$ is a regenerative process over $1 = \tau_0 < \tau_1 < \ldots$ with $\alpha = E(\tau_1 - \tau_0) < \infty$, and we let

$$X_n = \sum_{k=1}^{\tau_1 - 1} I(Y_k > n) \qquad \text{for } k \geq 1.$$

We also let ξ be a random measure such that $\xi(t)$ has stationary independent increments with

$$E(e^{-s\xi(1)}) = \exp - \int_0^\infty (1 - e^{-sx}) d\mu(x),$$

where μ is a measure on $(0, \infty)$ satisfying $\int_0^\infty \min\{1, x\} d\mu(x) < \infty$.

Theorem 4.5. For $b_n^{-1}\xi_n \circ a_n \to \xi \circ (\alpha c)^{-1}$ it is necessary and sufficient that the following hold.

(1) $a_n P(b_n^{-1} X_n \leq t) \to \mu(t)$ for all t with $\mu(\{t\}) = 0$.

(2) $\lim_{\varepsilon \to 0} \underline{\lim}_{n \to \infty} a_n b_n^{-1} EX_n^\varepsilon = \lim_{\varepsilon \to 0} \overline{\lim}_{n \to \infty} a_n b_n^{-1} EX_n^\varepsilon = 0$, where X_n^ε is X_n truncated at ε.

Remark 4.6. Note that $b_n^{-1}\xi_n \circ a_n$ converges in distribution to a compound Poisson process whose jumps occur according to a Poisson process with intensity $(\alpha c)^{-1}$ and whose jump sizes have distribution μ if and only if (1) holds with $\mu(0,\infty) = 1$. (The latter implies (2).)

Proof. Let

$$\zeta_n(t) = b_n^{-1} \sum_{k=1}^{[a_n t]} X_{nk}, \text{ where } X_{nk} = \sum_{\ell=\tau_{k-1}}^{\tau_k - 1} I(Y_\ell > n).$$

Conditions (1) and (2) are necessary and sufficient for $\zeta_n(1) \xrightarrow{\mathcal{D}} \xi(1)$, see [13, Theorem 6.1] which is a generalization of [4, p. 564]. Also $\zeta_n(1) \xrightarrow{\mathcal{D}} \xi(1)$ is equivalent to $\zeta_n \xrightarrow{\mathcal{D}} \xi$, see [5, p. 480]. In addition, $b_n^{-1}\xi_n \circ a_n \xrightarrow{\mathcal{D}} \xi \circ (\alpha c)^{-1}$ is equivalent to $b_n^{-1}\eta_n \circ a_n \xrightarrow{\mathcal{D}} \xi \circ \alpha^{-1}$ by Theorem 3.1. Recall that $\eta_n(t) = \sum_{k=1}^{[t]} I(Y_k > n)$. Thus to prove the theorem, it remains to show that

$$(4.1) \qquad b_n^{-1}\eta_n \circ a_n \xrightarrow{\mathcal{D}} \xi \circ \alpha^{-1} \iff \zeta_n \xrightarrow{\mathcal{D}} \xi.$$

To this end we shall use the representation

$$(4.2)\qquad b_n^{-1}\eta_n o a_n = \zeta_n o \gamma_n + (b_n^{-1}\eta_n o a_n - \zeta_n o \gamma_n),$$

where

$$\gamma_n(t) = a_n^{-1}\nu(a_n t) \text{ and } \nu(t) = \max\{k : \tau_k \leq t\}.$$

We shall also use the fact that

$$(4.3)\qquad \gamma_n \xrightarrow{\mathcal{D}} \alpha^{-1}\Lambda \text{ where } \Lambda \text{ is the Lebesque measure.}$$

This follows since $\tau_k - \tau_{k-1}$ $(k \geq 1)$ are independent with a common distribution and mean α, and $t^{-1}\nu(t) \to \alpha^{-1}$ a.s.

To prove (4.1) we first assume that $\zeta_n \xrightarrow{\mathcal{D}} \xi$. From this and (4.3), it follows by Lemma 2.1 that

$$(4.4)\qquad \zeta_n o \gamma_n \xrightarrow{\mathcal{D}} \xi o \alpha^{-1}\Lambda = \xi o \alpha^{-1},$$

and

$$(4.5)\qquad b_n^{-1}\eta_n o a_n - \zeta_n o \gamma_n \leq \zeta_n o(\gamma_n + a_n^{-1}) - \zeta_n o \gamma_n \xrightarrow{\mathcal{D}} 0.$$

(Here $\zeta_n o(\gamma_n + a_n^{-1})(t) = \zeta_n(\gamma_n(t) + a_n^{-1})$.) Then from (4.2) - (4.5) it follows that

$$b_n^{-1}\eta_n o a_n \xrightarrow{\mathcal{D}} \xi o \alpha^{-1}.$$

Now assume the latter holds. Clearly

$$\zeta_n = \zeta_n o \gamma_n o \tilde{\gamma}_n = (b_n^{-1}\eta_n o a_n o \phi_n) o \tilde{\gamma}_n$$

where $\phi_n(t) = a_n^{-1}\tau_{\nu(a_n t)}$ and

$$\tilde{\gamma}_n(t) = \inf\{u : \gamma_n(u) > t\} = a_n^{-1}\tau_{[a_n t]}.$$

The latter follows by standard properties of inverses. Under our assumption we have

$$(b_n^{-1}\eta_n o a_n, \phi_n, \tilde{\gamma}_n) \xrightarrow{\mathcal{D}} (\xi o \alpha^{-1}, \Lambda, \alpha\Lambda).$$

Then by Lemma 2.1

$$\zeta_n \xrightarrow{\mathcal{D}} \xi o \alpha^{-1} o \alpha\Lambda = \xi.$$

This completes the proof.

We now consider semi-stationary thinnings. Namely, we assume that Y_k is a semi-stationary process over the integer-valued random variables $1 = \tau_0 < \tau_1 < \tau_2 \ldots$ This means that the sequence of vectors

$$(\tau_k - \tau_{k-1}, Y_{\tau_{k-1}}, \ldots, Y_{\tau_k - 2}, Y_{\tau_k - 1}, 0, 0, \ldots) \qquad \text{for } k \geq 1$$

is strictly stationary, see [27]. Special cases of these processes are stationary processes, positive recurrent Markov chains, regenerative processes and many combinations of these types of processes.

As an example suppose that the point at site S_k has a generic makeup described by β_k and the thinning operation at S_k is based on a parameter γ_k such that

$$Y_k = f(\beta_k, \gamma_k)$$

where f is some function. Suppose β_k is a positive recurrent Markov chain on a countable state space and $1 = \tau_0 < \tau_1 < \ldots$ are the successive indices k for which $\beta_k = i$. And suppose that γ_k is a stationary sequence which is independent of β_k. Then Y_k is semi-stationary over the τ's.

The next result on semi-stationary thinnings contains Theorems 4.1 and 4.4 as special cases.

Theorem 4.7. Suppose $Y_1, Y_2, \ldots$ is semi-stationary over $1 = \tau_0 < \tau_1 < \ldots$ with $\alpha = (E\tau_1 - \tau_0) < \infty$ such that $X_{nk} = \sum_{\ell = \tau_{k-1}}^{\tau_k - 1} I(Y_\ell > n)$ satisfies Conditions 4.3 and

$$a_n P(X_{n1} = 1) \to \lambda > 0, \text{ and } a_n P(X_{n1} \geq 2) \to 0.$$

Then $\xi_n \circ a_n \xrightarrow{D} \xi$ where ξ is a Poisson process with intensity $\lambda(\alpha c)^{-1}$.

Proof. Let

$$\zeta_n(t) = \sum_{k=1}^{[a_n t]} X_{nk} = \zeta_n'(t) + (\zeta_n(t) - \zeta_n'(t)),$$

where

$$\zeta_n'(t) = \sum_{k=1}^{[a_n t]} I(X_{nk} \geq 1).$$

From [17, Theorem 3.2] it follows that $\zeta_n' \xrightarrow{\mathcal{D}} \xi$. Also for each t

$$P(\zeta_n(t) - \zeta_n'(t) = 0) = P(\bigcap_{k=1}^{[a_n t]} \{X_{nk} \leq 1\})$$

$$= 1 - P(\bigcup_{k=1}^{[a_n t]} \{X_{nk} \geq 2\}) \geq 1 - [a_n t]P(X_{nk} \geq 2) \to 1.$$

That is, $\zeta_n - \zeta_n' \xrightarrow{\mathcal{D}} 0$. We then have $\zeta_n \xrightarrow{\mathcal{D}} \xi$. From this it follows, by an argument as in the second and third paragraphs in the last proof, that $\eta_n o a_n \xrightarrow{\mathcal{D}} \xi o c$. But this is equivalent to $\xi_n o a_n \xrightarrow{\mathcal{D}} \xi$ by Theorem 3.1.

5. Independent Thinning in General Spaces.

In this section we present two analogues to Theorem 4.1 for independent thinning in a locally compact second countable Hausdorff space X such as R^n. The Theorems 4.4, 4.5 and 4.7 do not have such clear-cut analogues, since the notions of regeneration and semi-stationarity require a total ordering of the initial points which can be specified in many ways.

For our first result, we assume that points are placed in X at random sites $S_1, S_2, \ldots$ and are deleted over time such that their lifetimes $Y_1, Y_2, \ldots$ are independent and identically distributed and are independent of the S's. At time n the thinned process is

$$\xi_n = \sum_k I(Y_k > n)\delta_{S_k}.$$

This ξ_n on X has the same structure as before. We also let

$$\zeta_n = \sum_k \delta_{a_n^{-1} S_k} \quad \text{and} \quad \xi_n o a_n(A) = \xi_n\{a_n x : x \in A\}.$$

where $a_n \to \infty$.

Theorem 5.1. If $a_n^{-1}\zeta_n \xrightarrow{\mathcal{D}} \mu$, where μ is a measure on X, and $a_n P(Y_1 > n) \to \lambda > 0$, then $\xi_n \circ a_n \xrightarrow{\mathcal{D}} \xi$, where ξ is a Poisson process with intensity measure $\lambda\mu$.

Proof. For any bounded disjoint Borel sets $A_1,\ldots,A_k$ in X we have,

$$(\xi_n \circ a_n(A_1),\ldots,\xi_n \circ a_n(A_k)) \overset{\mathcal{D}}{=} (\eta_{n1}(\zeta_n(A_1)),\ldots,\mu_{nk}(\zeta_n(A_k)))$$

where $\eta_{n1},\ldots,\eta_{nk}$ are independent copies of $\eta_n = \sum_k I(Y_k > n)\delta_k$ and are also independent of ζ_n. Under the assumptions

$$(\eta_{n1} \circ a_n,\ldots,\eta_{nk} \circ a_n) \xrightarrow{\mathcal{D}} (\eta_1,\ldots,\eta_k)$$

where $\eta_1,\ldots,\eta_k$ are independent Poisson processes with intensity λ (a constant). We also have

$$(a_n^{-1}\zeta_n(A_1),\ldots,a_n^{-1}\zeta_n(A_k)) \xrightarrow{\mathcal{D}} (\mu(A_1),\ldots,\mu(A_k)).$$

Then it follows by Lemma 2.1 that

$$(\xi_n \circ a_n(A_1),\ldots,\xi_n \circ a_n(A_k)) \xrightarrow{\mathcal{D}} (\eta_1(\mu(A_1)),\ldots,\eta_k(\mu(A_k)) \overset{\mathcal{D}}{=} (\xi(A_1),\ldots,\xi(A_k)).$$

This completes the proof.

The next result is a slight variation of the proceeding and its proof is the same. For this we assume that

$$\xi_n = \sum_k I(Y_k > n)\delta_{S_{nk}} \quad \text{and} \quad \zeta_n = \sum_k \delta_{S_{nk}}$$

where S_{nk} is the site of the k-th point at time n and the Y's are as above. This describes thinnings in which the points may move over time. We also let ζ be a random measure on X.

Theorem 5.2. If $a_n^{-1}\zeta_n \xrightarrow{\mathcal{D}} \zeta$ and $a_n P(Y_1 > n) \to \lambda > 0$, then $\xi_n \xrightarrow{\mathcal{D}} \xi$, where ξ is a conditional Poisson process with random intensity measure $\lambda\zeta$.

The conditional (or doubly stochastic) Poisson process ξ with intensity $\lambda\zeta$ is defined by

$$P(\xi(A_1) = n_1,\ldots,\xi(A_k) = n_k)$$
$$= \int\ldots\int u_1^{n_1}\ldots u_k^{n_k} e^{-u_1}\ldots e^{-u_k}/(n_1!\ldots n_k!)P(\lambda\zeta(A_1)\varepsilon du,\ldots, \lambda\zeta(A_k)\varepsilon du_k).$$

These processes were first studied by Cox and Lewis, see [19] and [26] and their references.

Theorem 5.2 for the Renyi thinning, where $P(Y_1 > n) = p^n$, was proved in [19] and in [12].

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